Detecting Phase Transitions in Intermittent Systems by Using the Thermodynamical Formalism*

Zoltán Toroczkai[a] and Áron Péntek[b]
Institute for Theoretical Physics, Eötvös University, Puskin utca 6–7, H-1088 Budapest, Hungary

Z. Naturforsch. **49a**, 1235–1237 (1994); received October 12, 1994

Here we illustrate the effectiveness of the thermodynamical formalism applied to a well known chaotic phenomenon, the intermittency. This leads us to a new classification for intermittent phenomena from the point of view of the generated chaotic phases in the spectrum of the generalized entropies $K(q)$. New types of intermittencies are found related to the absence or presence of phase transitions with infinite jump in $K(q)$. This is underlined with examples. It is also shown via examples that the existence of a marginally stable fixed point in the system is neither necessary nor sufficient for intermittency.

In the following we report on the investigation of intermittency by means of the thermodynamical formalism. More precisely we give criteria for the existence of nonanalyticies (phase transitions) in the spectrum of dynamical entropies $K(q)$ related to intermittent systems. Because we are interested in phase transitions in $K(q)$ it is useful to introduce the following phases [1]: 1. the chaotic chaos phase (CCP) corresponding to the region in q where $K(q)$ is nonzero and finite, 2. the regular chaos phase (RCP), where $K(q)$ is zero, and 3. the anomalous chaos phase (ACP), which is characterized by infinite Rényi entropies.

Generally, to build up the thermodynamics for a specific (chaotic) system one needs:

1. an encoding of the chaotic signal using a finite set of symbols (e.g. in intermittent systems one uses two finite subsets of symbols $\mathscr{E}_1$ and $\mathscr{E}_2$, standing for laminar and turbulent states, respectively);
2. the probability of a certain symbol sequence of finite length $p(\{s_i\}_n)$ and construct the partition sum
$$Z_n(q) = \sum_{\{s_i\}_n} [p(\{s_i\}_n)]^q; \quad (1)$$
3. the behaviour of $Z_n(q)$ in the thermodynamic limit $(n \to \infty)$ to extract $K(q)$ as
$$Z_n(q) \sim e^{P(q)n}. \quad (2)$$

Here $P(q) = K(q)(1-q)$ is the "pressure function" with respect to $p\{s_i\}_n$ and q is referred to in the literature as the 'inverse temperature'.

Usually, a system is considered to be intermittent if it produces signals with long laminar sequences and with an asymptotic power law decay in its time correlation. We take [3] a somewhat looser definition of intermittency, by considering a signal intermittent if it *looks* intermittent, i.e., long uniform blocks show up frequently in the symbolic codes of the dynamics independently of the form of the correlation decay. This means that the probability distribution of laminar lengths (PDLL) takes on relatively large values for short and intermediate lengths but the asymptotic form at large lengths is not necessarily of power-law type. Usually the power-law type behaviour is a consequence of the fact that besides a marginally stable fixed point a random reinjection is assumed.

A careful analysis based on an extended version of the statistical-mechanical formalism developed by Sato and Honda [2] led us to give [3] an exact criterion for the existence of phase transitions in intermittent systems in terms of explicitly measurable quantities from time series. Let us define the probability distribution of laminar lengths l for a symbol s in the form
$$p_s(l) = c_s\, e^{-W_s(l)} \quad (3)$$
with c_s some constant and $W_s(l)$ a nonnegative, asymptotically increasing function with the length l of

---

* Presented at the Workshop on Thermodynamic Formalism, Lavin (Engadin), Switzerland, April 23–26, 1994.

[a] Present address: Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, USA.
[b] Present address: Institute for Pure and Applied Physical Sciences, University of California at San Diego, La Jolla, California 92093-0075, USA.

Reprint requests to Z. Toroczkai.

0932-0784 / 94 / 1200-1235 $ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72027 Tübingen

the laminar blocks. The results derived in [3] can be summarized as follows:

(*i*) There is a phase transition point at $q=1$, and $P(q)=0$ for $q>1$ whenever $\beta(j) \equiv \frac{W(j)}{j} \to 0$ for at least one of the symbols (CCP–RCP transition);

(*ii*) When $\beta(j) \to \alpha\,(\alpha>0$, and finite) for all the symbols, the whole entropy function is smooth, and no phase transition occurs.

(*iii*) There is another phase transition at $q=0$, and $P(q)=\infty$ for $q<0$, whenever $\beta(j) \to \infty$ for at least one of the symbols (ACP–CCP transition).

Since in the RCP and ACP phases, where $P(q)$ is identically zero and infinite, respectively, this quantity is no longer appropriate to characterize these phases. However, one can derive the asymptotic expression for the truncated pressure function $P_k(q)$ which gives the limiting behaviour $P_k(q) \to P(q)=0$ and $P_k(q) \to P(q)=\infty$, respectively, in both phases, as

$$P_k(q) \sim -q\,\beta(k), \qquad k \gg 1. \tag{4}$$

Here k is the longest laminar length taken into consideration. The form (4) allows us to give the scaling for the partition function also:

$$Z_n(q) \sim e^{-qW(n)}, \qquad n \gg 1 \tag{5}$$

in these phases. Comparing (5) with (2) one can observe that the scaling is no longer exponential with respect to n (the volume) but is with respect to q (the inverse temperature). So, one could say that nonhyperbolicity can be regarded as a hyperbolic behaviour if one makes the rather 'strange' exchange between the quantities q and n. In this case, the important role is taken by the function $W(n)$ instead for the pressure function $P(q)$.

It is interesting that the behaviours (*i*), (*ii*), and (*iii*) can equally hold for systems generating intermittent signals, where the symbol in question is the one characterising the laminar phase. The case (*i*), when $\beta(j) \to 0$ (as $j \to \infty$), we call *Classical Intermittency*, because intermittencies characterized by a power-law decay belong to this class ($W(j) \sim \ln(j)$, and $\beta(j) \sim \ln(j)/j \to \infty$). For case (*ii*) we use the term *Borderline Intermittency* because the set of $\beta(j)$ functions, which are the asymptotically linear ones, is of measure-zero in the set of the possible $\beta(j)$ functions, and stands as a limiting case between the other two situations where phase transitions occur. Finally, we refer to case (*iii*) as *Anomalous Intermittency*, i.e. when one encounters divergent entropies.

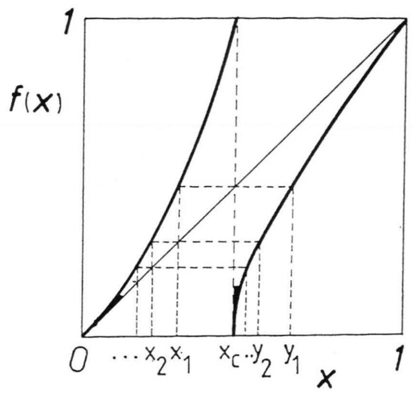

Fig. 1. Lorentz type map with a singular reinjecting branch. In spite of the marginally stable fixed point in the origin, the system will not generate an intermittent signal.

In the following we show that the existence of a marginally stable fixed point is neither a necessary nor a sufficient condition for intermittency. This is so because the reinjecting part of the dynamics can play an essential role if it contains singularities which 'compete' with the part of the system which generates the laminar behaviour. An example, is the map in Figure 1. The point x_c is mapped in one step onto the origin which is supposed to be marginally stable. Therefore the observed PDLL (or $W_0(j)$) should strongly depend on the specific form of the right branch of the map around $x=x_c$ (or $y=0$, in terms of the inverse functions $F_0(y)$ and $F_1(y)$). If the asymptotic behaviour of the function β_0 is

$$\beta_0(j) = \gamma j^{\tau-1}, \tag{6}$$

(including all the three cases (i), (ii) and (iii), by taking $\tau<1$, $\tau=1$ and $\tau>1$, respectively) then one obtains (see [3]) for the form of $F_1(y)$ around $y=0$

$$F_1(y) = x_c + C_2 \left(\frac{1}{y}\right)^{1-\tau} e^{-\delta(1/y)^{\tau}}, \tag{7}$$

where $\delta = \gamma\,[f''(0)]^{-\tau}$ (the origin is marginally stable, $f'(0)=1$). The phase transition caused by the marginally stable fixed point when a random reinjection were assumed, can be cancelled by a rather singular form of the branch $F_1(y)$ around $y=0$, namely

$$F_1(y) = x_c + C_2\, e^{-\delta(1/y)}. \tag{8}$$

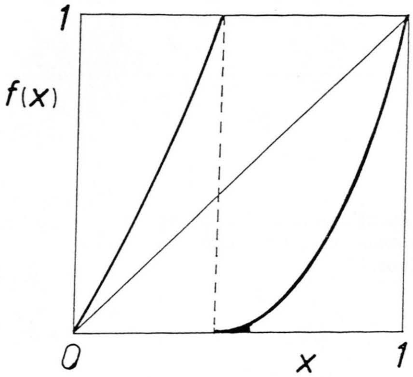

Fig. 2. Lorentz type map with a strongly singular reinjecting branch and without marginally stable fixed point. The generated time signal is intermittent.

Then, the signal looks intermittent if (8) holds in a very small region $y \in [0, \varepsilon)$, so that the intermittent behaviour is 'screened' only in a close vicinity $[0, F_0(\varepsilon))$ of the origin (the preimage of $[0, \varepsilon)$ with respect to the left branch, i.e. F_0^{-1}). A narrow channel (of width ε) formed by $F_0(y)$ and the first bisector still exists and is responsible for the intermittent signal. In the case when ε is not too small, the right branch cancels completely the intermittent effect of the origin, so we would not observe any intermittency, i.e. the existence of a marginally stable fixed point is not a sufficient conditions.

If one takes $\tau > 1$, the singularity becomes even stronger and due to (6) and (*iii*) we will have an ACP–CCP phase transition at $q = 1$, but with an intermittent signal. If $\tau < 1$ we encounter the usual intermittent situation with CCP–RCP transition.

Finally, we give an example which shows that to have an intermittent signal (output) the existence of a marginally fixed point is not a necessary conditions, see Figure 2 [4]. If the form of the right branch is fairly singular [3] (in order to get the asymptotic dependence (6)):

$$F_1(y) = x_c + C_1 \left(\ln\left(\frac{1}{y}\right)\right)^{1-\tau} e^{-\omega (\ln 1/y)^\tau}, \qquad (9)$$

an RCP–CCP phase transition occurs at $0 < \tau < 1$ and the signal looks intermittent (here $\omega = \gamma [f'(0) - 1]^{-\tau}$), in spite of the absence of the *marginally stable* fixed point.

This work was partially supported by the Hungarian Science Foundation under Grant Nos. OTKA 2090 and T4439 and by the Foundation for Hungarian Higher Education and Research.

[1] A. Csordás and P. Szépfalusy, Phys. Rev. A **39,** 4767 (1989).
[2] S. Sato and K. Honda, Phys. Rev. A **42,** 3233 (1990).
[3] Z. Toroczkai and Á. Péntek, Phys. Rev. E **48,** 136 (1993).
[4] J. Bene and P. Szépfalusy, Phys. Rev. A **46,** 801 (1992).